Universal Tomonaga-Luttinger liquid phases in one-dimensional strongly attractive $SU(N)$ fermionic cold atoms

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Abstract

A simple set of algebraic equations is derived for the exact low-temperature thermodynamics of one-dimensional multi-component strongly attractive fermionic atoms with enlarged $SU(N)$ spin symmetry and Zeeman splitting. Universal multi-component Tomonaga-Luttinger liquid (TLL) phases are thus determined. For linear Zeeman splitting, the physics of the gapless phase at low temperatures belongs to the universality class of a two-component asymmetric TLL corresponding to spin-neutral N-atom composites and spin-$(N-1)/2$ single atoms. The equation of states is also obtained to open up the study of multi-component TLL phases in 1D systems of N-component Fermi gases with population imbalance.

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A significant feature of one-dimensional (1D) many-body systems is the universal low energy physics of the gapless phase described by a Tomonaga-Luttinger liquid (TLL) [1], as recently revealed in experimental measurements on the thermodynamics of spin ladder materials [2]. In such systems magnetic fields drive phase transitions between gapped and gapless phases. In the gapless phase spin excitations (spinons) carrying spin-1/2 give rise to universal TLL thermodynamics.

On the other hand, exquisite control over the effective spin-spin interaction between cold atoms [3] has provided a new opportunity to rigorously test spin liquid behaviour via trapped fermionic atoms with higher spin symmetry [4, 5, 6, 7]. In particular, fermionic alkaline-earth atoms display an exact $SU(N)$ spin symmetry with $N = 2I + 1$ where I is the nuclear spin [8]. For example, $I = 9/2$ for ^{87}Sr and $I = 5/2$ for ^{171}Yb. Such fermionic systems with enlarged $SU(N)$ spin symmetry are expected to display a remarkable diversity of new quantum phases due to the existence of multiple charge bound states.

We derive the universal thermodynamics of the gapless phase in strongly attractive 1D fermions with N-component hyperfine states. We thus find that population imbalances controlled by Zeeman splitting can be used to explore and control multi-component TLL physics in 1D interacting fermionic gases with $SU(N)$ spin symmetry. The universal crossover from the regime of a relativistic multi-component Luttinger liquid to a nonrelativistic quantum critical regime is determined from the specific heat phase diagrams.

In principle, the thermodynamic Bethe ansatz (TBA) can provide the exact thermodynamics of such systems. However, the TBA involves an infinite number of coupled nonlinear integral equations which hinders access to the thermodynamics from both the numerical and analytical points of view [9, 10]. To overcome this obstacle, a method was proposed to analytically obtain thermodynamic quantities of the two-component attractive Fermi gas [11]. We build on this approach to reduce the multiple charge bound state TBA for attractive 1D fermions with arbitrary $SU(N)$ spin symmetry to a set of simple algebraic equations to provide universal and exact low-temperature thermodynamics of the model.

The model. We consider a system of N_f interacting fermions of equal mass m which may occupy N possible hyperfine levels ($|i\rangle$, $i = 1, \ldots, N$) labeled by N isospin states and constrained by periodic boundary conditions to a line of length L. The Hamiltonian [12, 13]

is

$$\mathcal{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N_f}\frac{\partial^2}{\partial x_i^2} + g_{1\mathrm{D}}\sum_{1\le i<j\le N_f}\delta(x_i - x_j) + E_z \tag{1}$$

with Zeeman energy $E_z = \sum_{i=1}^{N} N_f^i \epsilon_Z^i(\mu_B^i, B)$. Here N_f^i is the number of fermions in state $|i\rangle$ with Zeeman energy ϵ_Z^i (or say, a chemical potential μ_i) determined by the magnetic moments μ_B^i and the magnetic field B. The spin-independent contact interaction $g_{1\mathrm{D}}$ remains between fermions with different hyperfine states so that the spins are conserved, i.e., N_f^i with $i = 1, \ldots, N$ are good quantum numbers. The coupling constant $g_{1\mathrm{D}} = \hbar^2 c/m$ with interaction strength $c = -2/a_{1\mathrm{D}}$ determined by the effective 1D scattering length $a_{1\mathrm{D}}$. For simplicity, we choose the dimensionless units of $\hbar = 2m = 1$ for numerical calculation and use the dimensionless coupling constant $\gamma = c/n$ with linear density $n = N_f/L$. The Hamiltonian (1) exhibits $U(1) \times SU(N)$ symmetry where $U(1)$ is associated with the charge degree of freedom and $SU(N)$ with the N hyperfine spin states.

Ground states. The energy eigenspectrum is given in terms of the fermion quasimomenta $\{k_i\}$ satisfying the Bethe ansatz equations (BAE) [12, 13] by $E = \sum_{j=1}^{N_f} k_j^2$. For attractive interaction, the BAE allow charge bound states and spin strings. The $SU(N)$ symmetry acquires $N-1$ kinds of charge bound states which can be viewed as composites of r-atoms with total spin s, where $s = r(N-r)/2$ and $r = 2, \ldots, N$. In order to simplify calculations in the study of magnetism, we rewrite the Zeeman energy as $E_Z = -\sum_{i=1}^{N-1} H_i N_i$ where the independent parameters H_i with $i = 1, \ldots, N-1$ characterize the chemical potentials for N_1 unpaired fermions and N_i charge bound states of i-atoms. Here we set up explicit relations among the H_i's and arbitrary (nonlinear) Zeeman splittings $\Delta_{i+1\,i} = \epsilon_Z^{i+1} - \epsilon_Z^i$ via the relations $\Delta_{i+1\,i} = -H_{i-1} + 2H_i - H_{i+1}$ with $H_{N+1} = 0$. We shall show that equally spaced (linear) Zeeman splitting, i.e., $\Delta_{a+1\,a} = H$ for $a = 1, \ldots, N-1$, drives a smooth phase transition from bound states of N-atoms into a normal Fermi liquid at zero temperature, see Fig. 1(A). Nonlinear Zeeman splittings may trigger spin-s charge bound states as illustrated in Fig. 1(B), where the magnetic moments of paired states is $s = N-2$ [4].

For strong attraction ($|\gamma| \gg 1$) these charge bound states are stable and the system is strongly correlated. The corresponding binding energies of the charge bound states are given by $\epsilon_r = \hbar^2 c^2 r(r^2-1)/(24m)$. Solving the BAE with strongly attractive interaction, the ground state energy E_0^∞ per unit length in the thermodynamic limit is given explicitly

by (in units of $\hbar^2/2m$)

$$E_0^\infty \approx \sum_{r=1}^{N} \frac{\pi^2 n_r^3}{3r}\left(1+\frac{2}{|c|}A_r+\frac{3}{c^2}A_r^2\right)-\sum_{r=2}^{N} n_r\epsilon_r. \tag{2}$$

Here $n_r = N_r/L$ and we have defined the functions

$$A_r = \sum_{j=1}^{r-1}\sum_{i=j}^{N}\frac{4n_i\theta(r-2)}{r(i+r-2j)}+\sum_{i=r+1}^{N}\frac{4n_i\theta(N-r-1)}{r(i-r)}, \tag{3}$$

where $\theta(x)$ is the step function.

The general result (2) for arbitrary N unifies and extends known results [14, 15, 16] for isospin $S = 1/2, 1$ and $3/2$. Furthermore, we find that the low-lying excitations are described by the linear dispersion relations $\epsilon^r(k) = v_r(k - k_F^r)$, where $v_r \approx \frac{\hbar\pi n_r}{mr}(1+\frac{2}{|c|}A_r+\frac{3}{c^2}A_r^2)$ with $r = 1, \ldots, N$ are the velocities for unpaired fermions and charge bound states of r-atoms. We denote the corresponding Fermi momentum by k_F^r. These dispersion relations lead naturally to the universal form [17, 18] for finite-size corrections to the ground state energy

$$E(L, N_f) - LE_0^\infty \approx -\frac{\pi\hbar C}{6L}\sum_{r=1}^{N} v_r. \tag{4}$$

The central charge $C = 1$ for $U(1)$ symmetry. Here the universal finite-size corrections (4) indicate the TLL signature of 1D many-body physics [1].

Quantum phase transitions. At finite temperatures, the equilibrium states are determined by the minimization of the Gibbs free energy [20], which gives rise to a set of coupled nonlinear integral equations – the TBA equations [21]. The TBA equations provide not only finite temperature thermodynamics but also the band fillings with respect to Zeeman splittings and chemical potentials which may be conveniently used to analyze quantum phase transitions at zero temperature. From the TBA equations, we find that complete phase diagrams at zero temperature are determined by the independent external field-energy transfer relations

$$H_r = \frac{1}{12}r(N^2-r^2)c^2 + r(\mu^{(r)}-\mu^N), \tag{5}$$

for $r = 1, \ldots, N-1$. The effective chemical potentials $\mu^{(r)} = \mu + \frac{H_r}{r} + \frac{(r^2-1)}{3}\frac{c^2}{4}$ with $r = 1, \ldots, N$ are given by

$$\frac{\mu^{(r)}}{\pi^2} \approx \frac{n_r^2}{r^2}\left(1+\frac{2A_r}{|c|}+\frac{3A_r^2}{c^2}\right)+\frac{\vec{B}_r\cdot\vec{I}}{|c|}+\frac{3\vec{B}_r\cdot\vec{A}}{c^2}, \tag{6}$$

characterizing Fermi surfaces of stable charge bound states and unpaired fermions. Here $\vec{A} = (A_1, \ldots, A_N)$ and $\vec{B}_r = (B_r^1, \ldots, B_r^N)$ with

$$B_r^\ell = \frac{8n_\ell^3}{3r\ell}\left[\sum_{j=1}^{\ell-1}\frac{\theta(r-j)}{\ell(r+\ell-2j)} + \frac{\theta(r-\ell-1)}{\ell(r-\ell)}\right] \tag{7}$$

for $r, \ell = 1, \ldots, N$. $\vec{I}$ is a unit vector. The energy transfer relations (5) can be used to show that linear Zeeman splitting can only lift $SU(N)$ symmetry to $U(1)^N$ symmetry, recall Fig. 1.

For pure Zeeman splitting, if the external field H is less than a lower critical field H_{c1}, a molecular superfluid phase of spin-neutral bound states forms the ground state. We find the value

$$\begin{aligned} H_{c1} \approx\ & \frac{(N+1)n^2\gamma^2}{6} - \frac{2\pi^2 n^2}{N^4(N-1)}\left[1 + \frac{16Q}{3N^2|\gamma|}\right. \\ & \left. -\frac{8}{3N(N-1)|\gamma|} + \frac{20Q^2}{\gamma^2 N^4} - \frac{16Q}{\gamma^2 N^3(N-1)}\right] \end{aligned} \tag{8}$$

at which the spin gap is diminished. Here $Q = \sum_{j=1}^{N-1}\frac{1}{N-j}$. On the other hand, when $H > H_{c2}$, where

$$H_{c2} \approx \frac{(N+1)n^2\gamma^2}{6} + \frac{2\pi^2 n^2}{(N-1)}\left[1 - \frac{8}{3N(N-1)|\gamma|}\right] \tag{9}$$

the system is fully-polarized into a normal Fermi liquid. For the intermediate regime $H_{c1} < H < H_{c2}$ spin-neutral bound states of N-atoms and unpaired fermions coexist. The magnetization gradually increases from $m^z = 0$ to its normalized value $m^z = 1$ as the field increases from H_{c1} to H_{c2} (see the solid line Fig. 2). More subtle phases may be explored using the energy-transfer relations (5) by controlling the nonlinear Zeeman splitting parameters, e.g., the BCS pairing phase (see Fig. 1) by setting $\Delta_{2\,1} = h_1$ and $\Delta_{a+1\,a} = h_2$ for $a = 2, N-1$. All phase transitions are of second order with a linear field-dependent magnetization in the vicinity of critical points due to the condition of fixed total particle number.

Universal thermodynamics. At high temperatures $T \sim \epsilon_r$ (setting the Boltzmann constant to unity), thermal fluctuations can break the charge bound states while spin fluctuations lead to an effective ferromagnetic spin-spin interaction coupled to each Fermi sea of spin-s charge bound states and unpaired fermions. The effective ferromagnetic spin-spin coupling constants are given by $J^{(r)} = \frac{2}{r|c|}p^{(r)}$ for $r = 1, 2, \ldots, N-1$ [21]. In this sense, we may

simply view the non-neutral charge bound state as a molecule with spin s, which could flip its spin antiparallel to the external fields H_r due to thermal fluctuations. However, such spin fluctuations coupled to the channels of unpaired fermions and the spin-s charge bound states are suppressed by large fields H_r at low temperatures. Thus the low energy physics is dominated by charge density fluctuations. Indeed we show that in the physically interesting regime $T \ll \epsilon_r,\ \Delta_{i+1\,i},\ \gamma \gg 1$ the breaking of charge bound states and spin wave fluctuations is strongly suppressed. Thus each dressed energy can be written in a single particle form $\epsilon^r(k) = \hbar^2 r k^2/2m - \bar{\mu}^{(r)} + O(1/\gamma^3)$, where the marginal scattering energies among composites and unpaired fermions and spin-wave thermal fluctuations are considered in the chemical potentials $\bar{\mu}^{(r)}$.

The thermodynamics at finite temperatures can thus be obtained from the set of algebraic equations [21]

$$\begin{aligned} \bar{\mu}^{(r)} &\approx r\mu^{(r)} - \sum_{j=1}^{r} \sum_{\substack{i=j \\ i \neq 2j-i}}^{N} \frac{4P^{(i)}}{i(i+r-2j)|c|} + f_s^{(r)}, \\ p^{(r)} &\approx -\sqrt{\frac{rm}{2\pi\hbar^2}}\, T^{\frac{3}{2}}\, \mathrm{Li}_{\frac{3}{2}}\left(-e^{\bar{\mu}^{(r)}/T}\right) \end{aligned} \tag{10}$$

for $r = 1, \ldots, N$. Here $\mathrm{Li}_s(x)$ is the standard polylogarithm function and $f_s^{(r)}$ denotes the free energy of an effective ferromagnetic spin-spin interaction with coupling constant $J^{(r)}$ in the channel of the spin-s charge bound state [21]. However, because it is a spin singlet, there is no such effective spin-spin interaction for the spin-neutral bound state of N-atoms, i.e., $f_s^{(N)} = 0$ and $H_N = 0$. $p^{(r)}$ is the pressure for charge bound states of r-atoms. The pressure for the system is given by $p = \sum_{r=1}^{N} p^{(r)}$. The spin string contributions to thermal fluctuations in the physically interesting regime can be asymptotically calculated from the spin string equations for these non-spin-neutral charge bound states and unpaired fermions, i.e., $f_s^{(r)} \approx T e^{-\frac{\Delta_{r+1\,r}}{T}} e^{-\frac{J^{(r)}}{T}} I_0(\frac{J^{(r)}}{T})$, where $I_0(x) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}(\frac{x}{2})^{2k}$. It is important to observe that $f_s^{(r)}$ becomes exponentially small as $T \to 0$.

The suppression of spin fluctuations leads to a universality class of a multi-component TLL in each gapless phase, where the charge bound states of r-atoms form hard-core composite particles. In order to see this universal TLL physics, we calculate the leading low temperature corrections to the free energy by using Sommerfeld expansion with the TBA

equations to give

$$f \approx f_0 - \frac{\pi T^2}{6\hbar} \sum_{r=1}^{N} \frac{1}{v_r}. \tag{11}$$

This result is consistent with the finite-size correction result (4). In the above equation $f_0 = E_0^{\infty} - \sum_{r=1}^{N-1} n_r H_r$. This result proves the existence of TLL phases in 1D gapped systems at low temperatures. Although there is no quantum phase transition in 1D many-body systems at finite temperatures due to thermal fluctuations, the existence of the TLL leads to a crossover from relativistic dispersion to nonrelativistic dispersion between different regimes at low temperatures [19]. Nevertheless, we find that such a field-induced multi-component TLL only lies in a small portion of Zeeman parameter space. Linear Zeeman splitting may result in a two-component Luttinger liquid in a large portion of Zeeman parameter space at low temperatures.

The thermodynamics of the gapless phase for the model (1) can be analytically calculated with linear Zeeman splitting, i.e. with $\Delta_{a+1\,a} = H$ for $a = 1, \ldots, N-1$. In the regime $H_{c1} < H < H_{c2}$ the TBA equations (10) reduce to two coupled equations for $\bar{\mu}^{(1)}$ and $\bar{\mu}^{(N)}$. The rest of the effective $\bar{\mu}^{(r)} = 0$. From these two equations, we can obtain the density $n = \partial p/\partial \mu$, magnetization $m^z = \partial p/\partial H$ and free energy $f = \mu n - p$ by iteration. Fig. 2 shows the magnetization for attractive three-component fermions with a pure Zeeman field at different temperatures. We see that the linear field-dependent phase transitions in the vicinity of the critical points are smeared.

The analytic results (10) and (11) indicate that the magnetization develops minima at the same temperatures as when the two-component TLL is broken. Indeed, a deviation from the linear temperature-dependent specific heat $c_v = \frac{\pi T}{3\hbar}\left(\frac{1}{v_1} + \frac{1}{v_N}\right)$ and the magnetization minima occur around the same crossover temperatures – see the white-diamond-lines in Fig. 3 where we show a contour plot of the specific heat for $SU(3)$ and $SU(4)$. The hard-core N-atom composite particles and single atoms form an asymmetric two-component TLL lying below the white-diamond-lines. The filled-black-circles which separate the different regimes are determined by the magnetization values $m^z = 0$ and $m^z = 1$. The peaks at the critical points are expected when the unpaired (bound state) band starts to fill (empty) [22].

To conclude, our analytic results for universal thermodynamics and quantum phase transitions provide a unified description of attractive 1D $SU(N)$ fermions in the presence of external fields. Our formulae also provide the essential equation of states for studying

trapped N-component Fermi gases with population imbalance. They thus pave the way for the study of multi-component TLL phases in 1D systems of ultracold fermionic atoms.

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[1] T. Giamarchi, *Quantum Physics in one dimension*, Oxford University Press, Oxford (2004)

[2] M. Klanjsek et al., Phys. Rev. Lett. **101**, 137207 (2008); Ch. Rüegg et al., Phys. Rev. Lett. **101**, 247202 (2008)

[3] T. B. Ottenstein, T. Lompe, M. Kohnen, A. N. Wenz and S. Jochim, Phys. Rev. Lett. **101**, 203202 (2008); J. H. Huchans et al., Phys. Rev. Lett. **102**, 165302 (2009); J. R. Williams et al., Phys. Rev. Lett. **103**, 130404 (2009)

[4] T.-L. Ho and S. Yip, Phys. Rev. Lett. **82**, 247 (1999)

[5] Á. Rapp, G. Zaránd, C. Honerkamp and W. Hofstetter, Phys. Rev. Lett. **98**, 160405 (2007); C. Honerkamp and W. Hofstetter, Phys. Rev. Lett. **92**, 170403 (2004)

[6] P. Lecheminant, E. Boulat and P. Azaria, Phys. Rev. Lett. **95**, 240402 (2005); S. Capponi et al., Phys. Rev. A, **77**, 013624 (2008)

[7] C. Wu, J.-P. Hu and S.-C. Zhang, Phys. Rev. Lett. **91**, 186402 (2003); D. Controzzi and A. M. Tsvelik, Phys. Rev. Lett. **96**, 097205 (2006)

[8] A. V. Gorshkov, et al, arXiv:0905.2610; M. A. Cazalilla, A. F. Ho and M. Ueda, New J. Phys. **11**, 103033 (2009)

[9] P. Kakashvili and C. J. Bolech, Phys. Rev. A **79**, 041603(R) (2009)

[10] M. Casula, D. M. Ceperley and E. J. Mueller, Phys. Rev. A **78**, 033607 (2008)

[11] E. Zhao, X.-W. Guan, W. V. Liu, M. T. Batchelor and M. Oshikawa, Phys. Rev. Lett. **103**, 140404 (2009)

[12] B. Sutherland, Phys. Rev. Lett. **20**, 98 (1968)

[13] M. Takahashi, Prog. Theor. Phys. **44**, 899 (1970)

[14] X.-W. Guan, M. T. Batchelor, C. Lee and M. Bortz, Phys. Rev. B **76**, 085120 (2007); X.-W. Guan, M. T. Batchelor, C. Lee and H.Q. Zhou, Phys. Rev. Lett. **100**, 200401 (2008); X.-W. Guan, M. T. Batchelor, J. Y. Lee and C. Lee, EPL **86**, 50003 (2009)

[15] T. Iida and M. Wadati, J. Phys. Soc. Jpn, **77**, 024006 (2008)

[16] X.-J. Liu, H. Hu and P. D. Drummond, Phys. Rev. A **77**, 013622 (2008)

[17] H. W. J. Blöte, J. L. Cardy and M. P. Nightingale, Phys. Rev. Lett. **56**, 742 (1986)

[18] I. Affleck, Phys. Rev. Lett. **56**, 746 (1986)

[19] Y. Maeda, C. Hotta and M. Oshikawa, Phys. Rev. Lett. **99**, 057205 (2007)

[20] M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models*, Cambridge University Press, Cambridge, (1999)

[21] The details of the derivation of the TBA equations and the result (10) will be presented elsewhere.

[22] T. Vekua, Phys. Rev. B **80**, 201402(R) (2009)

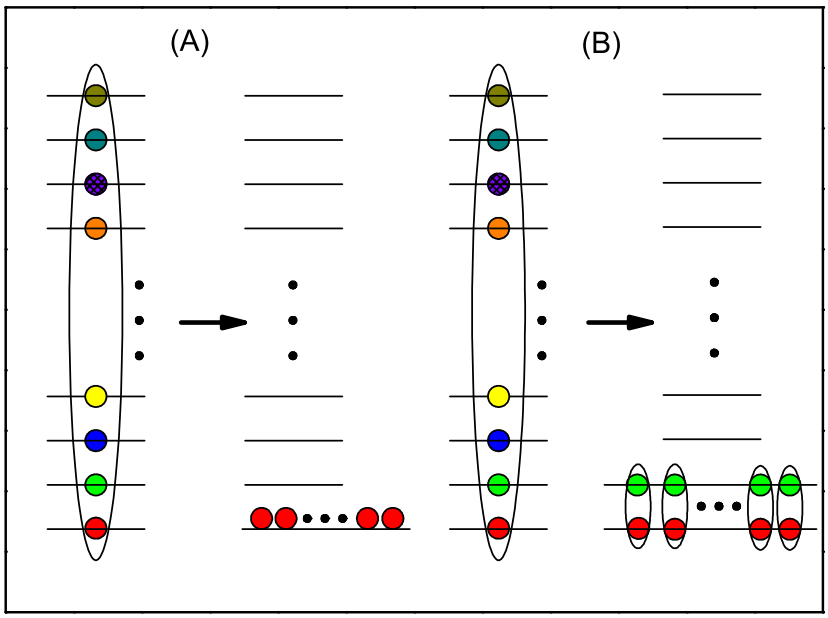

FIG. 1: Phase transitions from the bound states of N-fermions into (A) normal Fermi liquid and (B) paired states are induced by linear and nonlinear Zeeman splittings, respectively. Ellipses denote the charge bound states.

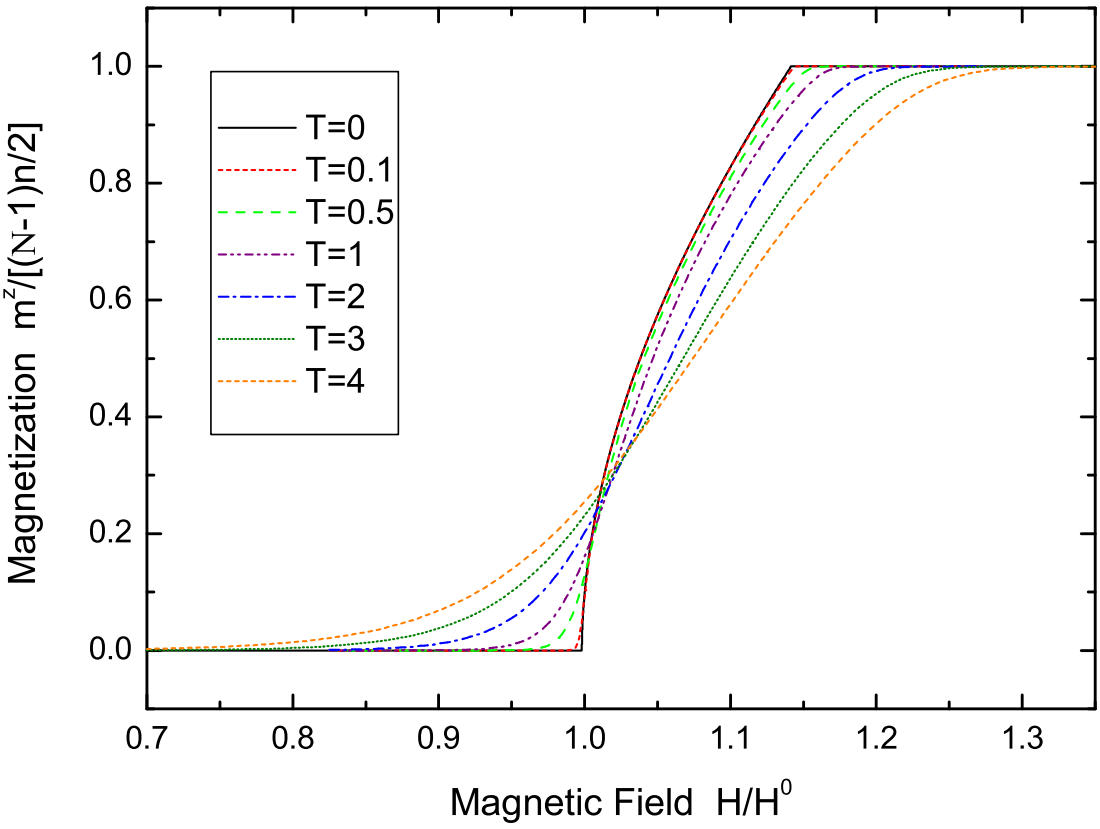

FIG. 2: Typical magnetization vs external field for $SU(3)$ fermions with strong coupling ($c = -10$ and $n = 1$) at different temperatures. The curves for $SU(N)$ are similar.

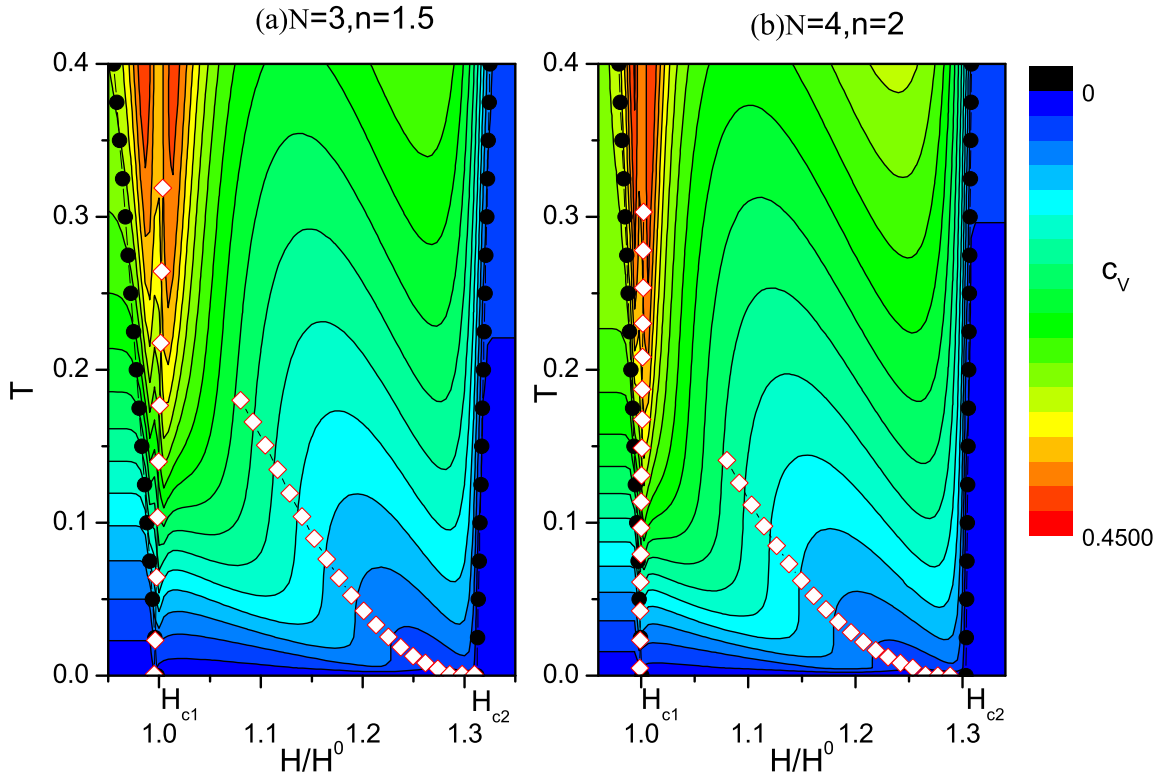

FIG. 3: Universal low temperature phase diagram: specific heat in the T-H plane for $SU(3)$ and $SU(4)$. H is rescaled by $H^0 = 2\epsilon_N/(N-1)$. An asymmetric two-component TLL remains within the regime between $H_{c1} < H < H_{c2}$ below the white-diamond-lines. TLL of spin-neutral bound states of N-atoms and TTL of fully-polarized fermionic atoms lie below the left and right filled black-circles, respectively.